18.06 MIDTERM 3

December 6, 2019 (50 minutes)

Please turn cell phones off completely and put them away.

No books, notes, or electronic devices are permitted during this exam.

You must show your work to receive credit. **JUSTIFY EVERYTHING**.

Please write your name on **ALL** pages that you want graded (those will be the ones we scan).

The back sides of the paper will **NOT** be graded (for scratch work only).

Do not unstaple the exam, nor reorder the sheets.

Problem 1 has $\boxed{3}$ parts, Problem 2 has $\boxed{5}$ parts, Problem 3 has $\boxed{5}$ parts.

NAME: ______________________________

MIT ID NUMBER: ______________________________

RECITATION INSTRUCTOR: ______________________________

PROBLEM 1 **NAME:** ____________________

(1) Choose three real numbers a, b, c such that the matrices:

$$A = \begin{bmatrix} 1 & a \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & c \\ 1 & 1 \end{bmatrix}$$

have the properties that:

- A has two distinct real eigenvalues
- B has two identical real eigenvalues (i.e. a repeated eigenvalue)
- C has two complex (non-real) eigenvalues

In each of these three cases, compute the eigenvalues in question. Show your work. *(15 pts)*

Solution: We compute the characteristic polynomial for these matrices to get

$$P_A(\lambda) = \lambda^2 - 2\lambda + (1 - a)$$

and similarly for the other matrices.
Thus we get the eigenvalues are given by

$$\lambda_{1/2} = 1 \pm \sqrt{a}$$

Thus to get 2 distinct real eigenvalues we need $a > 0$.
To get two identical eigenvalues we need $b = 0$.
And to two complex eigenvalues we need $c < 0$.
(2) Diagonalize A (the matrix with distinct real eigenvalues from part (1)), i.e. write it as:

$$A = VDV^{-1}$$

where V is an invertible 2×2 matrix and D is a diagonal 2×2 matrix. Explain your reasoning in figuring out V and D, and detail the step-by-step process. *(10 pts)*

Solution: We need to compute the eigenvectors for the matrix A above. To do this we need to compute the nullspaces of the following matrices

$$\begin{bmatrix} \mp\sqrt{a} & a \\ 1 & \mp\sqrt{a} \end{bmatrix}$$

So the eigenvectors for $\lambda_{1/2} = 1 \pm \sqrt{a}$ is given by $\begin{bmatrix} \pm\sqrt{a} \\ 1 \end{bmatrix}$.
Thus we get the diagonalization is given by V the matrix of eigenvectors and D the diagonal matrix of eigenvalues. And so we get the factorisation

$$A = \begin{bmatrix} \sqrt{a} & -\sqrt{a} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 + \sqrt{a} & 0 \\ 0 & 1 - \sqrt{a} \end{bmatrix} \frac{1}{2} \begin{bmatrix} \frac{1}{\sqrt{a}} & 1 \\ -\frac{1}{\sqrt{a}} & 1 \end{bmatrix}$$

(3) Recall that B has a repeated eigenvalue λ (which you should have computed in part (1)).

- Compute the eigenspace of λ, i.e. the subspace of vectors $\mathbf{v} \in \mathbb{R}^2$ such that $B\mathbf{v} = \lambda\mathbf{v}$.
- Use this to compute the geometric multiplicity of λ.
- Is B diagonalizable? *(10 pts)*

Solution: Note from part (1), we have $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and the only eigenvalue is 1. SO to compute the eigenspace we need to compute the nullspace of

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

So the eigenspace is spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
It follows from the above that the geometric multiplicity of the eigenvalue $\lambda = 1$ is 1.
On the other hand the algebraic multiplicity of $\lambda = 1$ is 2, so the algebraic and geometric multiplicities are not equal. We thus do not have a basis of eigenvectors and hence the matrix is not diagonalizable.

PROBLEM 2
Throughout this problem, the matrix A has the following Singular Value Decomposition:

$$A = \underbrace{\frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{3}\begin{bmatrix} 1 & x & 2 \\ 2 & 2 & y \\ 2 & -1 & -2 \end{bmatrix}}_{V^T}$$

where the matrices U and V are orthogonal and x, y denote two mystery real numbers.

(The matrices U and V include the prefactors $\frac{1}{5}$ and $\frac{1}{3}$, so the top-left entry of U is $\frac{3}{5}$ and the top-left entry of V is $\frac{1}{3}$. Recall that orthogonal means that $U^T U = I_2$ and $V^T V = I_3$)

(1) What are the values of x, y, based on the information provided? Explain how you know. *(5 pts)*

Solution: Note that the rows of V hence the columns of V^T are orthogonal, since V is an orthogonal matrix.
We thus need x and y to satisfy the equations

$$x + 2 * 2 + 2 * (-1) = 0$$
$$2 + 2y + 2 * (-2) = 0$$

Thus we get $x = -2$ and $y = 1$.

(2) Fill in the blanks (no explanation needed):

- the rank of the matrix A is $\underline{2}$ *(5 pts)*

- the eigenvalues of A^TA are $\underline{0,\ 1 \text{ and } 4}$, and those of AA^T are $\underline{1 \text{ and } 4}$ *(5 pts)*

- an eigenvector of A^TA is $\underline{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}$ *(5 pts)*

Hint: the answers to the blanks above are all encoded in the SVD of A

Solution: The reason for the above is the rank of a matrix is given by the number of singular values.
The eigenvalues of A^TA and AA^T are given by the squares of the singular values as well as 0's in order to have the correct number of eigenvalues.
The SVD of A gives precisely a diagonalization of A^TA and AA^T, with matrices of eigenvectors given by U and V respectively, so the eigenvectors are given by the columns of these matrices respectively.

(3) Write A as a sum of two rank 1 matrices (it suffices to write these rank 1 matrices as <u>a column times a scalar times a row</u>, e.g. $\mathbf{u} \cdot \sigma \cdot \mathbf{v}^T$, you don't need to explicitly multiply the column, scalar and row out). *(5 pts)*

Solution: We can rewrite the given SVD as the following sum

$$A = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix} * 2 * \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} * 1 * \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}$$

And so this gives the require decomposition of A into a sum of two rank 1 matrices.

(4) Compute the pseudo-inverse A^+ of A, and explain how you got it (your answer for A^+ should be a 3×2 matrix with explicit numbers as entries). *(5 pts)*

Solution: To compute the pseudo inverse from the SVD, we need the formula $A^+ = V\Sigma^+U^T$, where Σ^+ is just given by taking the inverse of the singular values. We thus get the formula

$$A^+ = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 19 & 8 \\ 10 & 20 \\ 14 & -2 \end{bmatrix}$$

(5) Use A^+ to compute a least squares solution to $A\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (i.e. you must find a vector $\mathbf{v} \in \mathbb{R}^3$ such that $A\mathbf{v}$ is as close as possible to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; explain which formula you are using).

(5 pts)

Solution: To solve this, we recall a solution to least squares using the pseudo-inverse, is given by

$$v^+ = A^+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 9 \\ 10 \\ 4 \end{bmatrix}$$

PROBLEM 3

(1) Compute the eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ of the following matrix:

$$E = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

In this problem only, you are allowed to guess the eigenvalues and eigenvectors without going through the whole process of working them out, since they are quite simple. *(10 pts)*

Solution: You are allowed to just guess the answer. But if you wanted to do it systematically, it would look something like this. We first compute the characteristic polynomial, to get

$$P_E(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$$

So the eigenvalues are given by $\lambda_1 = 0$ and $\lambda_2 = 2$.

The eigenvectors for λ_1 are given by the nullspace of E, so we get $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Similarly the eigenvectors of λ_2 are given by the nullspace of

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

and so we get $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(2) Fill in the blank: $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal because the matrix E is symmetric.

(5 pts)

Solution: Eigenvectors of different eigenvalues for a symmetric matrix are orthogonal.

For the remainder of this problem, consider the following setting: Alice and Bob run into Mr. Papadopoulos, who randomly chooses a letter among α, β, γ with equal probability.

- If Mr. Papadopoulos chooses α, then he gives Alice \$9 and Bob \$0
- If Mr. Papadopoulos chooses β, then he gives Alice \$0 and Bob \$9
- If Mr. Papadopoulos chooses γ, then he gives Alice \$3 and Bob \$6

Consider the random variables X_A (resp. X_B) = the amount of money Alice (resp. Bob) gets.

(3) Compute the expected values (a.k.a. the means) $E[X_A]$ and $E[X_B]$. *(5 pts)*

Solution: Put the two random variables in a vector $\mathbf{X} = \begin{bmatrix} X_A \\ X_B \end{bmatrix}$, whose possible values are:

$$\mathbf{x}_1 = \begin{bmatrix} 9 \\ 0 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 9 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

with probabilities $\frac{1}{3}$. The expected value of the random vector is then:

$$E[\mathbf{X}] = \mu = \frac{\mathbf{x}_1}{3} + \frac{\mathbf{x}_2}{3} + \frac{\mathbf{x}_3}{3} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

and the expected values of X_A and X_B are just the entries of the vector above.

(4) Compute the covariance matrix K of the random variables X_A and X_B. *(5 pts)*

Solution: The covariance matrix is given by:

$$K[(\mathbf{X}-\mu)(\mathbf{X}-\mu)^T] = K = \frac{1}{3} \cdot \begin{bmatrix} 5 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 5 & -5 \end{bmatrix} + \frac{1}{3} \cdot \begin{bmatrix} -4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -4 & 4 \end{bmatrix} + \frac{1}{3} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -14 \\ -14 & 14 \end{bmatrix}$$

(5) Harder: find two linear combinations of X_A and X_B (call these linear combinations Y and Z) such that the covariance of Y and Z is 0. What are the variances of Y and Z? *(5 pts)*

Solution: To do this we need to diagonalize the matrix K. But note that the matrix $K = 14E$, where E was studied in part (1). We thus already know the eigenvectors are $\mathbf{v}_1$ and $\mathbf{v}_2$, with eigenvalues $14\lambda_1 = 0$ and $14\lambda_2 = 28$.
Thus if we normalize $\mathbf{v}_1$ and $\mathbf{v}_2$, we get the diagonalization of K is given by

$$K = QDQ^T = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 28 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

And thus the independent random variables are given by

$$Q^T \begin{bmatrix} X_A \\ X_B \end{bmatrix} = \begin{bmatrix} \frac{X_A + X_B}{2} \\ \frac{X_A - X_B}{2} \end{bmatrix}$$

And their variances are given by the eigenvalues of K, ie are 0 and 28 respectively. Note that variance 0, just means it is constant and indeed the sum of X_A and X_B is always \$9.